

Linear Connections on the Quantum Plane

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Abstract: A general definition has been proposed recently of a linear connection and a metric in noncommutative geometry. It is shown that to within normalization there is a unique linear connection on the quantum plane and there is no metric.

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1 Linear connections

There have been several models proposed of noncommutative geometries (Connes 1986, Dubois-Violette 1988), some of which are based on quantum groups (Woronowicz 1987, Pusz & Woronowicz 1989, Wess & Zumino 1990, Maltiniotis 1993). A definition of a linear connection which uses only the left-module structure of the differential forms has been proposed by Chamseddine *et al* (1993). An algebra of differential forms has however a natural structure of a bimodule. Recently linear connections have been considered in the particular case of differential calculi based on derivations (Dubois-Violette & Michor 1994a,b). and more generally (Mourad 1994) which make essential use of this bimodule structure. We shall here apply the general definition to the particular case of the quantum plane. We shall show that to within normalization there is a unique linear connection on the quantum plane. The connection is not metric compatible. There is in fact no metric in the sense we have defined it.

We first recall the definition of a linear connection in commutative geometry, in a form (Koszul 1960) which allows for a noncommutative generalization. Let V be a differential manifold and let $(\Omega^*(V), d)$ be the ordinary differential calculus on V . Let H be a vector bundle over V associated to some principle bundle P . Let $\mathcal{C}(V)$ be the algebra of smooth functions on V and \mathcal{H} the left $\mathcal{C}(V)$ -module of smooth sections of H .

A connection on P is equivalent to a covariant derivative on H . This in turn can be characterized as a linear map

$$\mathcal{H} \xrightarrow{D} \Omega^1(V) \otimes_{\mathcal{C}(V)} \mathcal{H} \quad (1.1)$$

which satisfies the condition

$$D(f\psi) = df \otimes \psi + fD\psi \quad (1.2)$$

for arbitrary $f \in \mathcal{C}(V)$ and $\psi \in \mathcal{H}$. There is an immediate extension of D to a map

$$\Omega^*(V) \otimes_{\mathcal{C}(V)} \mathcal{H} \rightarrow \Omega^*(V) \otimes_{\mathcal{C}(V)} \mathcal{H}$$

by requiring that it be an antiderivation of degree 1. From (1.2) it follows that the difference between two covariant derivatives is an algebra morphism of \mathcal{H} .

The definition of a connection as a covariant derivative has an immediate extension (Connes 1986) to noncommutative geometry. Let \mathcal{A} be an arbitrary algebra and $(\Omega^*(\mathcal{A}), d)$ be a differential calculus over \mathcal{A} . One defines a covariant derivative on a left \mathcal{A} -module \mathcal{H} as a map

$$\mathcal{H} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes \mathcal{H} \quad (1.3)$$

which satisfies the condition (1.2) but with $f \in \mathcal{A}$. There is again an extension of D to a map

$$\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \rightarrow \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}$$

by requiring that it be an antiderivation of degree 1.

A linear connection on V can be defined as a connection on the cotangent bundle to V . It can be characterized as a linear map

$$\Omega^1(V) \xrightarrow{D} \Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V) \quad (1.4)$$

which satisfies the condition

$$D(f\xi) = df \otimes \xi + fD\xi \quad (1.5)$$

for arbitrary $f \in \mathcal{C}(V)$ and $\xi \in \Omega^1(V)$.

If, for simplicity, we suppose V to be parallelizable we can choose a globally defined moving frame θ^α on V . The connection form ω^α_β is defined then in terms of the covariant derivative of the moving frame:

$$D\theta^\alpha = -\omega^\alpha_\beta \otimes \theta^\beta. \quad (1.6)$$

Because of (1.2) the covariant derivative $D\xi$ of an arbitrary element $\xi = \xi_\alpha \theta^\alpha \in \Omega^1(V)$ can be written as $D\xi = (D\xi_\alpha) \otimes \theta^\alpha$ where

$$D\xi_\alpha = d\xi_\alpha - \omega^\beta_\alpha \xi_\beta. \quad (1.7)$$

Let π be the projection of $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ onto $\Omega^2(V)$ defined by the wedge product on the forms. The torsion form Θ^α can be defined as

$$\Theta^\alpha = (d - \pi D)\theta^\alpha. \quad (1.8)$$

The derivative $D_X \xi$ along the vector field X ,

$$D_X \xi = i_X D\xi, \quad (1.9)$$

is a linear map of $\Omega^1(V)$ into itself. In particular $D_X \theta^\alpha = -\omega^\alpha_\beta(X) \theta^\beta$. Using D_X an extension of D can be constructed to the tensor product $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$. We define

$$D_X(\theta^\alpha \otimes \theta^\beta) = D_X \theta^\alpha \otimes \theta^\beta + \theta^\alpha \otimes D_X \theta^\beta \quad (1.10)$$

Now let σ be the action on $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ defined by the permutation of two derivations:

$$\sigma(\xi \otimes \eta)(X, Y) = \xi \otimes \eta(Y, X) \quad (1.11)$$

and define $\sigma_{12} = \sigma \otimes 1$ as acting on the tensor product over $\mathcal{C}(V)$ of three factors of $\Omega^1(V)$. Then (1.10) can be rewritten without explicitly using the vector field as

$$D(\theta^\alpha \otimes \theta^\beta) = D\theta^\alpha \otimes \theta^\beta + \sigma_{12}(\theta^\alpha \otimes D\theta^\beta). \quad (1.12)$$

Define $\pi_{12} = \pi \otimes 1$. If the torsion vanishes one finds that

$$\pi_{12} D^2 \theta^\alpha = -\Omega^\alpha_\beta \otimes \theta^\beta \quad (1.13)$$

where $\Omega^\alpha{}_\beta$ is the curvature 2-form. Notice that the equality

$$\pi_{12}D^2(f\theta^\alpha) = f\pi_{12}D^2\theta^\alpha \quad (1.14)$$

is a consequence of the identity

$$\pi(\sigma + 1) = 0 \quad (1.15)$$

The module $\Omega^1(V)$ has a natural structure as a right $\mathcal{C}(V)$ -module and the corresponding condition equivalent to (1.5) is determined using the fact that $\mathcal{C}(V)$ is a commutative algebra:

$$D(\xi f) = D(f\xi). \quad (1.16)$$

Using σ this can also be written in the form

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f. \quad (1.17)$$

By extension, a linear connection over a general noncommutative algebra \mathcal{A} with a differential calculus $(\Omega^*(\mathcal{A}), d)$ can be defined as a linear map

$$\Omega^1 \xrightarrow{D} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \quad (1.18)$$

which satisfies the condition (1.5) for arbitrary $f \in \mathcal{A}$ and $\xi \in \Omega^1$. We have here set $\Omega^1(\mathcal{A}) = \Omega^1$. The module Ω^1 has again a natural structure as a right \mathcal{A} -module but in the noncommutative case it is impossible in general to consistently impose the condition (1.16) and a substitute must be found. We must decide how it is appropriate to define $D(\xi f)$ in terms of $D(\xi)$ and df . We propose (Mourad 1994) to postulate the existence of a map

$$\Omega^1 \otimes_{\mathcal{A}} \Omega^1 \xrightarrow{\sigma} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \quad (1.19)$$

which satisfies (1.15) to replace the one defined by (1.11). We define then $D(\xi f)$ by the equation (1.17) but using (1.19) instead of (1.11). The extension of D to $\Omega^1 \otimes \Omega^1$ is given by (1.12) but again using (1.19). In order that the two different ways of calculating $D(\xi fg)$ yield the same result we must impose that σ be right \mathcal{A} -linear. In general

$$\sigma^2 \neq 1. \quad (1.20)$$

A covariant derivative is a map of the form (1.18) which satisfies the Leibniz rules (1.5) and (1.17). Because of (1.15) the image of the operator $d - \pi D$ acting on $\Omega^1(\mathcal{A})$ is a bi-submodule Θ of $\Omega^2(\mathcal{A})$. It is the torsion submodule.

The extension of D to $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ is given by the analogue of (1.12):

$$D(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta). \quad (1.21)$$

We can then define the map

$$\Omega^1 \xrightarrow{\pi_{12}D^2} \Omega^2 \otimes_{\mathcal{A}} \Omega^1 \quad (1.22)$$

which can be extended by a projection

$$\Omega^2 \otimes_{\mathcal{A}} \Omega^1 \rightarrow (\Omega^2/\Theta) \otimes_{\mathcal{A}} \Omega^1. \quad (1.23)$$

After the projection, $\pi_{12}D^2$ is left \mathcal{A} -linear:

$$\pi_{12}D^2(f\xi) = f\pi_{12}D^2\xi. \quad (1.24)$$

It will not in general be right \mathcal{A} -linear. However in the particular case which we shall consider in the next section the right-module structure is completely determined by the left-module structure. There is a representation ρ of the algebra such that

$$\pi_{12}D^2(\xi f) = (\pi_{12}D^2\xi)\rho(f) \quad (1.25)$$

after the projection (1.23).

As a simple example we mention the universal calculus which has the property that $\pi = 1$. Therefore from (1.15) we see that $\sigma = -1$. If the torsion is to vanish the only possible covariant derivative then is the ordinary exterior derivative. Every torsion-free linear connection has vanishing curvature.

A metric g on V can be defined as a $\mathcal{C}(V)$ -linear, symmetric map of $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ into $\mathcal{C}(V)$. This definition makes sense if one replaces $\mathcal{C}(V)$ by an algebra \mathcal{A} and $\Omega^1(V)$ by $\Omega^1(\mathcal{A})$. By analogy with the commutative case we shall say that the covariant derivative (1.17) is metric if the following diagram is commutative:

$$\begin{array}{ccc} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 & \xrightarrow{D} & \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \\ g \downarrow & & \downarrow 1 \otimes g \\ \mathcal{A} & \xrightarrow{d} & \Omega^1 \end{array} \quad (1.26)$$

In general symmetry must be defined with respect to the map σ . We impose then on g the condition

$$g\sigma = g. \quad (1.27)$$

2 The quantum plane

In this section we apply our general prescription to the quantum plane (Manin 1989), which possesses a natural map σ . The algebra of forms $\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \Omega^2$ has 4 generators $x^i = (x, y)$ and $\xi^i = dx^i = (\xi, \eta)$ which satisfy the relations

$$\begin{aligned} xy &= qyx, & \xi^2 &= 0, & \eta^2 &= 0, & \eta\xi + q\xi\eta &= 0, \\ x\xi &= q^2\xi x, & x\eta &= q\eta x + (q^2 - 1)\xi y, & y\xi &= q\xi y, & y\eta &= q^2\eta y. \end{aligned} \quad (2.1)$$

These conditions that can be written in the form

$$x^i x^j - q^{-1} \hat{R}^{ij}_{kl} x^k x^l = 0, \quad (2.2a)$$

$$x^i \xi^j - q \hat{R}^{ij}_{kl} \xi^k x^l = 0, \quad (2.2b)$$

$$\xi^i \xi^j + q \hat{R}^{ij}_{kl} \xi^k \xi^l = 0. \quad (2.2c)$$

By grouping the indices the 4-index tensor \hat{R}^{ij}_{kl} can be written as a 4×4 matrix:

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \quad (2.3)$$

If the generators (a, b, c, d) of $SL_q(2, \mathbb{C})$ are written in the form of a matrix

$$a_j^i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.4)$$

then the invariance of Ω^* under the action of $SL_q(2, \mathbb{C})$ follows from the identity

$$\hat{R}^{ij}_{kl} a_m^k a_n^l = a_k^i a_l^j \hat{R}^{kl}_{mn}. \quad (2.5)$$

Introduce the trivial differential calculus on $SL_q(2, \mathbb{C})$ with the differential d given by

$$da_j^i = 0. \quad (2.6)$$

The result of the coaction of $SL_q(2, \mathbb{C})$ on x^i and ξ^i is then

$$x^{i'} = a_j^i \otimes x^j, \quad \xi^{i'} = a_j^i \otimes \xi^j. \quad (2.7)$$

and from (2.5) it follows that $x^{i'}$ and $\xi^{i'}$ satisfy the same relations as x^i and ξ^i .

We introduce the 1-form

$$\theta = x\eta - qy\xi. \quad (2.8)$$

It is easily seen that

$$\theta^2 = 0 \quad (2.9)$$

and that θ is invariant under the coaction of $SL_q(2, \mathbb{C})$:

$$\theta' = 1 \otimes \theta. \quad (2.10)$$

It is in fact, to within multiplication by a complex number, the only invariant element of Ω^1 . From (2.1) we deduce the commutation relations

$$x^i \theta = q \theta x^i, \quad \xi^i \theta = -q^{-3} \theta \xi^i. \quad (2.11)$$

To fix the definition of a covariant derivative we must first introduce the operator σ of Equation (1.19). If we take the covariant derivative of both sides of equation (2.2b) we find that σ is determined on $\xi^i \otimes \xi^j$. It is the inverse of the matrix $q\hat{R}$. Written out in detail it becomes

$$\begin{aligned} \sigma(\xi \otimes \xi) &= q^{-2} \xi \otimes \xi, & \sigma(\xi \otimes \eta) &= q^{-1} \eta \otimes \xi, \\ \sigma(\eta \otimes \xi) &= q^{-1} \xi \otimes \eta - (1 - q^{-2}) \eta \otimes \xi, & \sigma(\eta \otimes \eta) &= q^{-2} \eta \otimes \eta. \end{aligned} \quad (2.12)$$

The extension to $\Omega^1 \otimes_{\Omega^0} \Omega^1$ is given by the right Ω^0 -linearity. In fact, in this case σ is also left Ω^0 -linear. One verifies immediately that the condition (1.15) is satisfied. As a result of the linearity one finds

$$\begin{aligned} \sigma(\xi \otimes \theta) &= q^{-3} \theta \otimes \xi, & \sigma(\theta \otimes \xi) &= q \xi \otimes \theta - (1 - q^{-2}) \theta \otimes \xi, \\ \sigma(\eta \otimes \theta) &= q^{-3} \theta \otimes \eta, & \sigma(\theta \otimes \eta) &= q \eta \otimes \theta - (1 - q^{-2}) \theta \otimes \eta, \end{aligned}$$

as well as

$$\sigma(\theta \otimes \theta) = q^{-2} \theta \otimes \theta.$$

Although $\sigma^2 \neq 1$, one finds that σ satisfies the Hecke relation

$$(\sigma + 1)(\sigma - q^{-2}) = 0.$$

The eigenvectors are $\xi \otimes \xi$, $\eta \otimes \eta$ and $\xi \otimes \eta + q^{-1} \eta \otimes \xi$ corresponding to q^{-2} and $\xi \otimes \eta - q \eta \otimes \xi$ corresponding to -1 . The exterior algebra (symmetric algebra) is obtained by dividing the tensor algebra by the ideal generated by the eigenvectors of q^{-2} (the eigenvector of -1). The braid relation

$$\sigma_{12} \sigma_{23} \sigma_{12} = \sigma_{23} \sigma_{12} \sigma_{23}$$

assures us that on the tensor product of three copies of Ω^1 the two ways of taking the product yield the same answer. We shall however not explicitly use this fact.

There is a unique one-parameter family of covariant derivatives compatible with the algebraic structure (2.1) of Ω^* . It is given by

$$D\xi^i = \mu^4 x^i \theta \otimes \theta. \quad (2.13)$$

The parameter μ must have the dimensions of inverse length. From the invariance of θ it follows that D is invariant under the coaction of $SL_q(2, \mathbb{C})$. From Equation (2.9) one sees that the torsion vanishes.

Using (1.21) one finds the equality

$$\pi_{12}D^2\xi^i = \Omega^i \otimes \theta \quad (2.14)$$

where the 2-form Ω^i is given by

$$\Omega^i = \mu^4 q^{-2} (q^2 + 1)(q^4 + 1) x^i \xi \eta. \quad (2.15)$$

It vanishes for $q = \pm i$ and $q^2 = \pm i$ but it does not vanish when $q = 1$. There is a preferred family of non-trivial linear connections on the ordinary complex 2-plane which are stable under the quantum deformation. Equation (2.14) can also be written in the usual form analogous to (1.13)

$$\pi_{12}D^2\xi^i = -\Omega^i_j \otimes \xi^j, \quad (2.16)$$

with the curvature 2-form given by

$$\Omega^i_j = \mu^4 (1 + q^{-2})(1 + q^{-4}) \begin{pmatrix} q^2 xy & -qx^2 \\ q^2 y^2 & -xy \end{pmatrix} \xi \eta. \quad (2.17)$$

The operator $\pi_{12}D^2$ is a left-module morphism by construction. One finds that the representation ρ of (1.25) is given by $\rho(f)(x^i) = f(q^2 x^i)$. Since Ω^3 vanishes the Bianchi identities are trivially satisfied.

The metric is a Ω^0 -linear map from $\Omega^1 \otimes_{\Omega^0} \Omega^1$ into Ω^0 which satisfies (1.27). It is straightforward to see that there can be no metric. One can consistently impose the condition $g\sigma = -g$ but this map resembles rather a symplectic form. In the limit $q = 1$ there is a metric, the ordinary euclidean metric, but the connection (2.1.2) is not a metric connection. This can be seen from the absence of any symmetry in the matrix on the right-hand side of (2.17).

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